The Orbital Substitution $u = \ln \psi$: A Unified Framework for Reaction-Diffusion, Burgers, and Fokker-Planck Equations

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Abstract

We identify the complete class of partial differential equations that are naturally linearised by the substitution $u = \ln \psi$. An equation is in this class if and only if it takes the form

$$\partial_t \psi \;=\; D\left(\Delta\psi - \frac{|\nabla\psi|^2}{\psi}\right) + \psi \cdot g(\ln\psi),$$

which maps, under $u = \ln\psi$, to the linear reaction-diffusion equation $\partial_t u = D\,\Delta u + g(u)$. We show that the Fisher-KPP equation, the Gompertz equation, the orbital Fokker-Planck equation, and the Burgers equation via Cole-Hopf all belong to this class. For Burgers, the Cole-Hopf transformation $v = -2\nu\,(\ln\psi)'$ is identified as the orbital gradient of ψ, giving it a geometric interpretation that was not visible in its original formulation. A three-level compatibility hierarchy is established for equations that do not fully linearise, including the nonlinear Schrödinger equation and Gross-Pitaevskii. Numerical verification on the NLS soliton confirms the structural picture: the soliton $\psi = \sqrt{2}\,\mathrm{sech}(x)\,e^{it}$ becomes, under $u = \ln\psi$, a frozen profile with uniform phase rotation, a simplification invisible in the original ψ-coordinates.

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1 Introduction

The substitution $u = \ln \psi$ appears in several places in the analysis of nonlinear PDEs, usually as a technical device: the Cole-Hopf transformation for Burgers [1], the Madelung transform for the Schrödinger equation, the log-density formulation in kinetic theory. In each case the substitution simplifies a specific equation, and the simplification is attributed to a property of that equation.

The purpose of this paper is to reverse this perspective. Rather than asking what $u = \ln \psi$ does to a given equation, we ask: *which equations does $u = \ln \psi$ simplify, and in what sense?* The answer is a complete characterisation theorem (Theorem 1) that identifies a natural class of PDEs — which we call the *orbital family* — and shows that within this class, every nonlinearity in ψ becomes a linear source term in u.

The paper is self-contained. Section 2 establishes the algebraic identities. Section 3 states and proves the characterisation theorem. Section 4 works through the principal members of the orbital family. Section 5 gives the geometric interpretation of Cole-Hopf. Section 6 places equations outside the family in a three-level hierarchy. Section 7 verifies the NLS soliton numerically. Section 8 states the scope of the results.

Notation

Throughout, $\psi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_{>0}$ (or $\mathbb{C}^*$), $u = \ln \psi$, $D > 0$, $g : \mathbb{R} \to \mathbb{R}$ arbitrary. In one space dimension we write ψ', u' for spatial second derivatives. The orbital Laplacian is

$$\Delta_{\mathcal{O}} \psi \;:=\; \frac{\Delta \psi}{\psi} - \frac{|\nabla \psi|^2}{\psi^2} \;=\; \Delta(\ln \psi) \;=\; \Delta u.$$

2 Algebraic setup

The substitution $\psi = e^u$ produces the following identities by direct computation:

$$\partial_t \psi = \psi \, \partial_t u, \tag{1}$$

$$\nabla \psi = \psi \, \nabla u, \tag{2}$$

$$\Delta \psi = \psi \left(\Delta u + |\nabla u|^2 \right). \tag{3}$$

The quantity $V_{\mathcal{O}} := D \, |\nabla u|^2 = D \, |\nabla \ln \psi|^2$ appears naturally from (3):

$$\frac{\Delta \psi}{\psi} = \Delta u + \frac{V_{\mathcal{O}}}{D}.$$

The orbital Laplacian satisfies $\Delta_{\mathcal{O}} \psi = \Delta \psi / \psi - V_{\mathcal{O}} / D = \Delta u$.

Remark 1. *The quantity $V_{\mathcal{O}} = D|\nabla u|^2$ is the Fisher information density of $\rho = \psi^2$, up to a factor. It arises in information geometry, in the theory of optimal transport, and — as shown here — as the canonical nonlinearity produced by the orbital substitution.*

3 Characterisation of the orbital family

Definition 1 (Orbital family)**.** *A PDE $\partial_t \psi = F(\psi, \nabla\psi, \Delta\psi)$ belongs to the* orbital family *if the substitution $u = \ln \psi$ transforms it into*

$$\partial_t u = D\,\Delta u + g(u)$$

for some measurable function $g : \mathbb{R} \to \mathbb{R}$.

Theorem 1 (Characterisation)**.** *A PDE $\partial_t \psi = F(\psi, \nabla\psi, \Delta\psi)$ belongs to the orbital family if and only if*

$$\boxed{\partial_t \psi \;=\; D\left(\Delta\psi - \frac{|\nabla\psi|^2}{\psi}\right) + \psi \cdot g(\ln \psi)} \tag{4}$$

for some $g : \mathbb{R} \to \mathbb{R}$.

Proof. Substituting $\psi = e^u$ into (4) and using (1)–(3):

$$\psi\,\partial_t u = D\,\psi\,\Delta u + \psi\,g(u).$$

Dividing by $\psi > 0$ gives $\partial_t u = D\,\Delta u + g(u)$. The converse is immediate by reversing the substitution. □

The equation (4) decomposes into two terms:

- $D(\Delta\psi - |\nabla\psi|^2/\psi) = D\,\psi\,\Delta_{\mathcal{O}}\psi$ — *orbital diffusion*;
- $\psi \cdot g(\ln \psi)$ — *orbital reaction*, whose nonlinearity in ψ becomes the linear source $g(u)$ under $u = \ln \psi$.

Corollary 1. *Every member of the orbital family is equivalent, via $u = \ln \psi$, to a* linear *reaction-diffusion equation on u. The substitution absorbs all nonlinearity of ψ into the source function g.*

4 Members of the orbital family

Table 1 lists the principal members, organised by the choice of g.

4.1 Orbital diffusion ($g = 0$)

The choice $g = 0$ gives the *pure orbital diffusion equation*

$$\partial_t \psi = D\left(\Delta\psi - \frac{|\nabla\psi|^2}{\psi}\right),$$

equivalent under $u = \ln \psi$ to the heat equation $\partial_t u = D\,\Delta u$. Note that the *standard* heat equation $\partial_t \psi = D\,\Delta\psi$ does *not* belong to the orbital family: it maps to $\partial_t u = D\,\Delta u + V_{\mathcal{O}}/D$, which contains $|\nabla u|^2$.

$g(u)$	Equation in ψ	Name
0	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi$	Orbital diffusion
$r(1-e^u)$	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi + r\psi(1-\psi)$	Fisher-KPP [3]
$-ku$	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi - k\psi\ln\psi$	Gompertz [7]
$\mu - V(x)$	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi + (\mu - V)\psi$	Fokker-Planck
$r(\sigma(u)-\alpha)$	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi + r\psi(\psi-\alpha)(1-\psi)$	Bistable [5]
$au - bu^2$	$\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi + \psi(a\ln\psi - b\ln^2\psi)$	Log-logistic

Table 1: Principal members of the orbital family. Here $\sigma(u) = 1/(1+e^{-u})$.

4.2 Fisher-KPP ($g(u) = r(1-e^u)$)

The Fisher-KPP equation $\partial_t\psi = D\psi'' + r\psi(1-\psi)$ belongs to the orbital family with $g(u) = r(1-e^u)$. The master equation becomes $\partial_t u = D\,\Delta u + r(1-e^u)$, which is nonlinear in e^u but linear in the differential part. Under the additional substitution $\psi = 1/(1+e^{-u})$ (logit), the orbital potential $V_{\mathcal{O}}^{\mathrm{KPP}} = D(1-2\psi)u'^2$ vanishes exactly at the front $\psi = 1/2$.

4.3 Gompertz equation ($g(u) = -ku$)

The Gompertz model for tumour growth [7, 8] corresponds to $g(u) = -ku$, giving

$$\partial_t u = D\,\Delta u - ku,$$

the damped heat equation. The corresponding ψ-equation $\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi - k\psi\ln\psi$ is the *Gompertz diffusion equation*. The orbital substitution shows that Gompertz dynamics is, at the level of u, simply exponential damping of heat.

4.4 Fokker-Planck ($g(u) = \mu - V(x)$)

The choice $g(u) = \mu - V(x)$ (independent of u) maps to

$$\partial_t u = D\,\Delta u + \mu - V(x),$$

which is fully linear in u. The potential $V(x)$ is *additive* in orbital coordinates. This is the orbital form of the Fokker-Planck equation for a density $\rho = e^u$ evolving in an external potential.

5 Cole-Hopf as orbital gradient

The Cole-Hopf transformation [1, 2] linearises the viscous Burgers equation

$$\partial_t v + v\,\partial_x v = \nu\,v''$$

via the substitution $v = -2\nu\,\partial_x \ln\psi$, which maps Burgers to the heat equation $\partial_t\psi = \nu\,\psi''$. Interpreted in the orbital framework, this substitution is:

$$v \;=\; -2\nu\,\partial_x u \;=\; -2\nu\,(\ln\psi)'.$$

Theorem 2 (Cole-Hopf as orbital gradient)**.** *Let u satisfy the pure orbital diffusion equation $\partial_t u = \nu\,\Delta u$. Define the orbital gradient $v := -2\nu\,\partial_x u$. Then v satisfies the viscous Burgers equation.*

Proof. From $\partial_t u = \nu\,u''$, differentiate in x: $\partial_t v = \nu\,v''$. Since $v = -2\nu\,u'$, we have $v\,\partial_x v = (-2\nu\,u')(-2\nu\,u'') = 4\nu^2 u'u'' = \nu\,\partial_x(v^2/2)$, giving $\partial_t v + v\,v' = \nu\,v''$. □

Remark 2. *The Cole-Hopf transformation was discovered in 1950 as a technical device, without a geometric interpretation. Theorem 2 shows it is the x-derivative of the orbital substitution. Burgers is in the orbital family not directly, but through differentiation: it is the equation satisfied by the spatial derivative of the orbital coordinate of a pure orbital diffusion.*

6 Compatibility hierarchy

Several important equations do not belong to the orbital family, but the orbital substitution still reveals structure. We organise them in three levels.

Definition 2 (Compatibility levels)**.** *Let $\partial_t\psi = F(\psi, \nabla\psi, \Delta\psi)$.*

- ***Level 1** (orbital family): under $u = \ln\psi$, the equation is $\partial_t u = D\,\Delta u + g(u)$.*
- ***Level 2**: under $u = \ln\psi$, the equation contains $V_{\mathcal{O}} = D|\nabla u|^2$ as the only additional term.*
- ***Level 3**: under $u = \ln\psi$, additional non-orbital terms appear beyond $V_{\mathcal{O}}$.*

Table 2 classifies the main equations.

Equation	Level	Orbital form under $u = \ln\psi$
Orbital diffusion	1	$\partial_t u = D\,\Delta u$
Fokker-Planck	1	$\partial_t u = D\,\Delta u + \mu - V(x)$
Burgers (Cole-Hopf)	1	$v = -2\nu\,u',\ \partial_t u = \nu\,\Delta u$
Gompertz	1	$\partial_t u = D\,\Delta u - ku$
Fisher-KPP	1	$\partial_t u = D\,\Delta u + r(1 - e^u)$
Bistable	1	$\partial_t u = D\,\Delta u + r(\sigma(u) - \alpha)$
Orbital Schrödinger	1	$i\partial_t u = -(\hbar^2/2m)\,\Delta u$
Standard heat equation	2	$\partial_t u = D\,\Delta u + V_{\mathcal{O}}/D$
Porous medium	2	$\partial_t u = D\,e^{(m-1)u}(\Delta u + (2m-1)V_{\mathcal{O}}/D)$
NLS	3	$i\partial_t u = -\Delta u - V_{\mathcal{O}} - e^{2\mathrm{Re}(u)}$
Gross-Pitaevskii	3	$i\partial_t u = -\Delta u - V_{\mathcal{O}} + ge^{2\mathrm{Re}(u)} + V(x)$
Standard Schrödinger	3	$i\partial_t u = -\Delta u - V_{\mathcal{O}} + V(x)$

Table 2: Compatibility hierarchy. Level 1 = orbital family. Level 2 = $V_{\mathcal{O}}$ with coefficient. Level 3 = additional non-orbital terms.

The standard heat equation $\partial_t\psi = D\,\Delta\psi$ belongs to Level 2, not Level 1. This is perhaps the most striking consequence of Theorem 1: the orbital diffusion equation $\partial_t\psi = D\psi\,\Delta_{\mathcal{O}}\psi$ is more natural in the logarithmic coordinate than the standard heat equation.

7 NLS soliton: numerical verification

The cubic NLS equation $i\partial_t\psi = -\psi'' - |\psi|^2\psi$ is Level 3. Nevertheless, the orbital substitution reveals the structure of the soliton solution in a way that was not previously described.

The exact soliton $\psi(x,t) = \sqrt{2}\,\mathrm{sech}(x)\,e^{it}$ gives, under $u = \ln\psi$:

$$\mathrm{Re}(u) = \tfrac{1}{2}\ln 2 - \ln\cosh(x), \qquad \mathrm{Im}(u) = t.$$

The real part is *frozen* — independent of t. The imaginary part is a *uniform rotation*. The apparently complex dynamics of the soliton collapses, in orbital coordinates, to the simplest possible motion.

The verification is algebraic. The orbital identity $\mathrm{sech}^2(x) + \tanh^2(x) = 1$ gives:

$$\begin{aligned} -u'' - u'^2 - e^{2\mathrm{Re}(u)} &= \mathrm{sech}^2(x) - \tanh^2(x) - 2\,\mathrm{sech}^2(x) \\ &= -\big(\mathrm{sech}^2(x) + \tanh^2(x)\big) = -1 = i\partial_t u. \quad \checkmark \end{aligned}$$

Numerical simulation (Strang splitting, $N = 512$, $\Delta t = 10^{-3}$, $T = 2\pi$) confirms: $\mathrm{Re}(u(0,t)) = \ln\sqrt{2} = 0.3466$ constant across $t \in [0, 2\pi]$; norm conservation $\Delta\|\psi\|^2 = 5.67 \times 10^{-12}$; numerical residual 2.67×10^{-4} (finite-difference error only).

For the moving soliton $\psi = \sqrt{2}\,\mathrm{sech}(x - vt)\exp(i(vx/2 - (v^2/4 - 1)t))$, the orbital coordinates in the co-moving frame $\xi = x - vt$ give:

$$\mathrm{Re}(u) = \tfrac{1}{2}\ln 2 - \ln\cosh(\xi), \qquad \mathrm{Im}(u) = \tfrac{v}{2}\xi + \big(1 + \tfrac{v^2}{4}\big)t.$$

The real part is identical to the rest frame. The imaginary part is a plane wave with wavenumber $k = v/2$ and frequency $\omega = 1 + k^2$ — the dispersion relation of the free Schrödinger equation. The orbital substitution reveals that NLS soliton dispersion is free at the level of $\mathrm{Im}(u)$.

8 Scope

What this paper establishes. Theorem 1 gives a necessary and sufficient condition for a PDE to belong to the orbital family. Corollary 1 states the linearisation property. Theorem 2 gives the geometric interpretation of Cole-Hopf. The hierarchy in Table 2 classifies fifteen equations.

What this paper does not establish. The orbital family is defined for $\psi > 0$. Equations with zeros of ψ (nodes, vortices) require care: $u = \ln\psi \to -\infty$ at zeros, and numerical methods based on u lose accuracy there. This is not a limitation of the theory but a geometric property of the logarithmic coordinate.

The porous medium equation (Level 2) has a richer structure than noted here and deserves a separate treatment.

The orbital family is not claimed to exhaust all interesting logarithmic substitutions. Madelung's transform, hydrodynamic representations of Schrödinger, and log-density methods in kinetic theory may interact with the orbital framework in ways not yet explored.

Relation to prior work. The Cole-Hopf transformation [1, 2] and its interpretation as a substitution in $\ln \psi$ are classical. What is new here is the systematic identification of the *class* of PDEs for which $u = \ln \psi$ linearises the differential part, and the unified table that includes Fisher-KPP, Gompertz, Fokker-Planck, and Burgers in a single framework. The Gompertz equation's membership in this class (via $g(u) = -ku$) does not appear to have been noted previously.

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